

Matroid multiple cyclic exchange property

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ABSTRACT. We prove a new exchange property for bases of a matroid that generalizes the multiple symmetric exchange property. For every bases B_1, \dots, B_k of a matroid and a subset $A_1 \subset B_1$ there exist subsets $A_2 \subset B_2, \dots, A_k \subset B_k$ such that all sets $(B_i \setminus A_i) \cup A_{i-1}$ achieved by a cyclic shift of A_i 's by one are bases.

Base exchange properties is a central theme in matroid theory. A non-empty family \mathfrak{B} of subsets of a finite set E forms a set of bases of a matroid, just from the definition, if it satisfies the *exchange property*. That is, if for every $B_1, B_2 \in \mathfrak{B}$ and $e_1 \in B_1 \setminus B_2$ there exists $e_2 \in B_2 \setminus B_1$, such that $(B_1 \setminus e_1) \cup e_2 \in \mathfrak{B}$ (we refer the reader to [8] for a background of matroid theory).

In this case a stronger property holds. For every bases B_1, B_2 of a matroid and an element $e_1 \in B_1 \setminus B_2$ there exists an element $e_2 \in B_2 \setminus B_1$, such that both sets $(B_1 \setminus e_1) \cup e_2$ and $(B_2 \setminus e_2) \cup e_1$ are bases. It is the *symmetric exchange property* discovered by Brualdi [2] (White's conjecture on the toric ideal of a matroid is based on this property, see [5, 7, 9]).

Even more is true matroid bases – one can exchange symmetrically not only single elements, but also subsets. The *multiple symmetric exchange property* asserts that for every bases B_1, B_2 of a matroid and a subset $A_1 \subset B_1$ there exists a subset $A_2 \subset B_2$, such that both sets $(B_1 \setminus A_1) \cup A_2$ and $(B_2 \setminus A_2) \cup A_1$ are bases. It was proved by Green [3], and soon after by Woodall [10] (see [4] for more exchange properties, and [6] for easy proofs of them).

In this note we formulate and prove a new exchange property that generalizes the multiple symmetric exchange property.

THEOREM 1 (Multiple cyclic exchange property). *For every bases B_1, \dots, B_k of a matroid and a subset $A_1 \subset B_1$ there exist subsets $A_2 \subset B_2, \dots, A_k \subset B_k$ such that all sets $(B_i \setminus A_i) \cup A_{i-1}$ achieved by a cyclic shift of A_i 's by one are bases.*

Notice that for $k = 2$ we get exactly the multiple symmetric exchange property.

Key words and phrases. Matroid, base exchange property.

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PROOF. The proof uses a link between list coloring of a matroids and base exchange properties, see [6]. We consider a certain assignment of lists of colors to elements of the ground set, and show that there exists a proper (elements of the same color form an independent set) coloring from these lists. This implies a desired exchange property.

Let M be a matroid on a ground set E with rank function r . Suppose B_1, \dots, B_k are bases of M , and $A_1 \subset B_1$. We can assume that bases B_i are pairwise disjoint (since if they are not, then we can introduce parallel elements).

Let L assign list $\{1\}$ to elements of $B_1 \setminus A_1$, list $\{2\}$ to elements of A_1 , list $\{i, i+1\}$ to elements of B_i for $i = 2, \dots, k-1$, and list $\{1, k\}$ to elements of B_k . For every color i consider the set of elements C_i that have i on their list

$$C_1 = B_k \cup (B_1 \setminus A_1), C_2 = A_1 \cup B_2, C_3 = B_2 \cup B_3, \dots, C_k = B_{k-1} \cup B_k,$$

and the matroid M_i with rank function r_i , achieved by restricting M to the set C_i .

Observe that for every $A \subset B_1 \cup \dots \cup B_k$ inequality $r_1(A) + \dots + r_k(A) \geq |A|$ holds. Indeed, $r_1(A) + \dots + r_k(A) =$

$$r(A \cap (B_k \cup (B_1 \setminus A_1))) + r(A \cap (A_1 \cup B_2)) + r(A \cap (B_2 \cup B_3)) + \dots + r(A \cap (B_{k-1} \cup B_k))$$

using submodularity of rank function to the first and the last summand we obtain

$$\geq r(A \cap (B_{k-1} \cup B_k \cup (B_1 \setminus A_1))) + r(A \cap (A_1 \cup B_2)) + r(A \cap (B_2 \cup B_3)) + \dots + r(A \cap B_k) \geq$$

$$\geq r(A \cap (B_{k-1} \cup (B_1 \setminus A_1))) + r(A \cap (A_1 \cup B_2)) + r(A \cap (B_2 \cup B_3)) + \dots + r(A \cap B_k)$$

and inductively

$$\begin{aligned} &\geq r(A \cap (B_2 \cup (B_1 \setminus A_1))) + r(A \cap (A_1 \cup B_2)) + r(A \cap B_3) + \dots + r(A \cap B_k) \geq \\ &\geq r(A \cap B_1) + r(A \cap B_2) + r(A \cap B_3) + \dots + r(A \cap B_k) = |A \cap B_1| + \dots + |A \cap B_k| = |A|. \end{aligned}$$

By the matroid union theorem there exist sets D_1, \dots, D_k , each D_i independent in M_i , such that $D_1 \cup \dots \cup D_k = B_1 \cup \dots \cup B_k$. Observe that $A_1 = D_2 \cap B_1$. Define $A_i = D_{i+1} \cap B_i$ for $i = 2, \dots, k$. Then $(B_i \setminus A_i) \cup A_{i-1} = D_i$, so it is a basis. \square

For graphic matroids the single element cyclic exchange property (when $|A_1|=1$) was firstly proved by Bartłomiej Bosek [1] using cycle axioms.

Theorem 1 is optimal in a sense that for every $k \geq 3$ there exists a matroid of rank 3 and k bases B_1, \dots, B_k for which one can not have additionally that all sets $(B_i \setminus A_i) \cup A_{i-2}$ achieved by a cyclic shift of A_i 's by two are bases.

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